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Scalar-Tensor Theories and the Principle of Equivalence

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Abstract

We investigate the restrictions on scalar-tensor theories of gravitation implied by the assumptions: (i) the field equations are derivable from an action principle, (ii) units of mass length and time are defined by atomic standards, and (iii) the principle of equivalence holds whenever gravitational self-energy can be neglected. We show that in all these theories the presence of gravitational energy in a system leads to violations of the principle of equivalence.

The results of the Eotvos experiment may be interpreted to mean that different kinds of energy contribute to the inertial mass of a system the same amount as to the gravitational mass. The contributions to the inertial and gravitational masses of ordinary materials due to rest masses of particles, (nuclear) electrostatic energy, and nuclear energy, are equal to within 1 part in 10^4, 1 part in 10^6, and 1 part in 10^5, respectively (Schiff, 1959). Since the energies associated with the weak and gravitational interactions in pieces of ordinary materials are extremely small, the Eotvos experiments gives no direct evidence on how these energies contribute to the gravitational mass. However, if the rest mass energies of particles contain appreciable amounts of weak and gravitational self-energy, then the equality of inertial and gravitational mass of particles indicates that the weak and gravitational energies contribute in the normal way. Just how much each kind of energy contributes to the rest mass of a particle is not known (and perhaps not knowable) and therefore we can only say that if we have a theory in which some forms of energy contribute to the gravitational mass in the 'wrong' way, then the fact that different particles fall with the same acceleration will be a profound mystery. The scalar-tensor theory of Brans & Dicke (1961) is such a theory. It has been shown (Nordvedt, 1969; Ohanian, 1971) that in this theory the inertial and gravitational masses differ by a term which is of the order of the gravitational self-energy. The fact that a principle of equivalence for particles cannot be derived in this theory must be regarded as a serious defect. (In general relativity an exact principle of equivalence can be derived both in the classical and quantum case no matter what self-energies are present.) In view of this difficulty one might ask what other scalar-tensor theories can be constructed and whether the principle of equivalence holds exactly in any of them.

In Section 1 we look for the most general scalar-tensor theory with (i) field equations derivable from an action principle, (ii) units of mass, length and time defined by atomic standards and (iii) a principle of equivalence that applies to all forms of energy *except* gravitational energy. In Section 2 we show that in all these scalar-tensor theories the presence of gravitational self-energy necessarily leads to a violation of the principle of equivalence.

1. *Restrictions on Scalar-Tensor Theories*

We assume that

(i) *The gravitational field is described by a macroscopic, long-range tensor-field $g_{\mu\nu}$ and scalar field ϕ. The field equations are obtained by variation of an action integral which is invariant under general coordination transformations. The Lagrangian density contains first derivatives at most quadratically.*

This first hypothesis implies that the part of the Lagrangian density which does not contain the matter variables can be written as†

$$\frac{c^4}{16\pi}\left[f(\phi)R + \frac{\omega\phi_{,\mu}\phi^{,\mu}}{\phi} + 2\lambda(\phi)\right]\sqrt{(-g)} \qquad (1.1)$$

where $f(\phi)$ and $\lambda(\phi)$ are arbitrary functions of ϕ and ω is a constant. In general ω could be a function of ϕ, but it is always possible to introduce a transformation of variables $\phi' = \phi'(\phi)$ such that in terms of the new scalar field ϕ' the Lagrangian density has the form given by (1.1) with ω equal to a constant.

(ii) *Atomic standards are used to define units of mass, length and time.*

As unit of mass we take the mass of one particular kind of particle. Because of assumption (iii) (see below) it is not important which particle we choose. For simplicity we will use a neutral, massive spin-zero meson (e.g., the π^0 meson). As unit of velocity we take the velocity of light so that $c = 1$. This will give us a unit of time once we choose a unit of length. The use of an atomic standard of length necessarily involves quantum mechanics. For our unit of length we will use the Compton wavelength of the massive meson. This choice is (conceptually, if not experimentally) more convenient than the use of the Bohr radius or the wavelength of light emitted in a particular transition.

Fierz (1956) has emphasized the importance of (ii) for the physical interpretation of the Jordan gravitational theory. The following argument,

† We use a metric of signature $+---$.

which is essentially that of Fierz, shows that our choice of units implies that the 'free' massive meson moves along the geodesics of $g_{\mu\nu}$.

The most general action for the meson field Φ compatible with (i) and the absence of nongravitational interactions is

$$\int \tfrac{1}{2}[\hbar^2 f_1 \Phi_{,\mu}\Phi^{,\mu} - m^2 c^2 f_2 \Phi^2 + f_3 \Phi\Phi_{,\mu}\phi^{,\mu}]\sqrt{(-g)}\, d^4 x \tag{1.2}$$

where f_1, f_2, f_3 are functions of ϕ and $g_{\mu\nu}$ (f_2 may also depend on the derivatives of ϕ and $g_{\mu\nu}$). We can introduce a change of variables by

$$\bar{\Phi} = \sqrt{(f_1)}\,\Phi \tag{1.3}$$

Since

$$\int \bar{\Phi}\bar{\Phi}_{,\mu}\phi^{,\mu} h(\phi, g_{\mu\nu})\, d^4 x = -\tfrac{1}{2}\int \bar{\Phi}^2[\phi^{,\mu} h(\phi, g_{\mu\nu})]_{,\mu}\, d^4 x \tag{1.4}$$

we can then express (2.2) in the form

$$\int \tfrac{1}{2}[\hbar^2 \bar{\Phi}_{,\mu}\bar{\Phi}^{,\mu} - m^2 c^2 \bar{f}_2 \bar{\Phi}^2]\sqrt{(-g)}\, d^4 x \tag{1.5}$$

where $\bar{f}_2$ is some new function of ϕ and its derivatives. The field equation for Φ is (suppressing the bars)

$$\hbar^2 \Box\Phi + m^2 c^2 f_2 \Phi = 0 \tag{1.6}$$

where $\Box\Phi \equiv (-g)^{-1/2}\partial_\mu[g^{\mu\nu}(-g)^{1/2}\partial_\nu\Phi]$. Let us introduce a local geodesic reference frame by choosing the coordinates such that

$$g^{\mu\nu} = \delta^{\mu\nu} \tag{1.7}$$

where $\delta^{\mu\nu}$ is the Lorentz metric. As is well known in the vicinity of any one point it is always possible to choose coordinates such that (1.7) holds and such that the first derivatives of the tensor field are zero. We then have

$$\hbar^2 \partial^\mu \partial_\mu \Phi + m^2 c^2 f_2 \Phi = 0 \tag{1.8}$$

This shows that the meson moves as a free wave with a Compton wavelength

$$\frac{\hbar}{mc\sqrt{(f_2)}} \tag{1.9}$$

Since this is to be constant by the definition of our unit of length, we must have

$$f_2 = 1 \tag{1.10}$$

Equation (1.6) then reduces to

$$\hbar^2 \Box\Phi + m^2 c^2 \Phi = 0 \tag{1.11}$$

In the classical limit the orbits of the particles are obtained by taking $\Phi = \exp(iS/\hbar)$ in equation (1.11) and retaining only the terms of order zero in $\hbar$. This gives the Hamilton-Jacobi equation

$$-g^{\mu\nu}\,\partial_\mu S\,\partial_\nu S + m^2 c^2 = 0 \tag{1.12}$$

which shows that the particles move along the geodesics of $g_{\mu\nu}$.

(iii) *In a gravitational field all sufficiently small systems in which the gravitational self-energy can be neglected fall with the same acceleration.*

By 'sufficiently small' we mean that the system must be of a size which is small compared to the distance over which the external gravitational field varies appreciably; this implies that no tidal forces act on the system. For the purposes of hypothesis (iii) it will be assumed that the gravitational self-energy of the known 'elementary' particles can somehow be neglected.

Since we have shown that the Φ-particles move along the geodesics of $g_{\mu\nu}$, it follows from (iii) that all localized systems that do not contain gravitational self-energy move along these geodesics. The tensor $g_{\mu\nu}$ therefore plays the role of metric, not only as far as the Φ-particles are concerned but in general. The tensor $g_{\mu\nu}$ will be regarded as *the* metric of space-time.

We next show that there are serious restrictions on the possible direct interactions between matter and the ϕ field. This is not surprising: if the ϕ field is directly coupled to some material system, it would usually produce a deviation from geodesic motion for that system contradicting (iii). A precise and general argument showing this has been given by Bergmann (1968). The following is a somewhat more explicit calculation.

Suppose that matter is described by a Lagrangian density $\mathscr{L}_m$ which depends, among other things, on ϕ and derivatives of ϕ. The complete Lagrangian density is then (taking $c = 1$)

$$\frac{1}{16\pi}\left[f(\phi)\,R + \omega\frac{\phi_{,\mu}\phi^{,\mu}}{\phi} + 2\lambda(\phi)\right]\sqrt{(-g)} + \mathscr{L}_m \tag{1.13}$$

and the field equations for $g_{\mu\nu}$ and ϕ are, respectively

$$\begin{aligned}-R^{\mu\nu} + \tfrac{1}{2}g^{\mu\nu}R = 8\pi\frac{T^{\mu\nu}}{f} &+ \frac{\omega}{\phi f}(\phi^{,\mu}\phi^{,\nu} - \tfrac{1}{2}g^{\mu\nu}\phi_{,\alpha}\phi^{,\alpha})\\ &+ \frac{1}{f}(f^{,\mu;\nu} - g^{\mu\nu}\Box f) - \frac{\lambda}{f}g^{\mu\nu}\end{aligned} \tag{1.14}$$

$$\frac{2\omega}{\phi}\Box\phi - \frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\mu} - Rf' - 2\lambda' - 16\pi\left(\mathscr{L}_m{}' - \partial_\mu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\mu}}\right) = 0 \tag{1.15}$$

where $T^{\mu\nu} = -2(-g)^{-1/2}(\partial/\partial g_{\mu\nu})\mathscr{L}_m$ and the primes denote differentiation with respect to ϕ. Because of the identity $(-R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R)^{;\nu} = 0$, equation (1.14) gives us a conservation theorem for the tensor $T_{\mu\nu}$:

$$0 = \left[8\pi\frac{T_{\mu\nu}}{f} + \frac{\omega}{\phi f}(\phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) + \frac{1}{f}(\phi_{,\mu;\nu}f' + \phi_{,\mu}\phi_{,\nu}f'' \right.$$
$$\left. - g_{\mu\nu}f'\,\Box\phi - g_{\mu\nu}f''\,\phi_{,\alpha}\phi^{,\alpha}) - \frac{\lambda}{f}g_{\mu\nu}\right]^{;\nu} \tag{1.16}$$

Carrying out the indicated differentiations and using equation (1.15) one finds that (1.16) can be written as

$$T_{\mu\nu}{}^{;\nu} = -\phi_{,\mu}\left[\frac{\partial\mathscr{L}_m}{\partial\phi} - \partial_\nu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\nu}}\right] \tag{1.17}$$

This conservation law determines the equation of motion of the system. Suppose that the system finds itself in an external field $g_{\mu\nu}$ (we treat the system as a test particle and neglect the gravitational field produced by the system). If we go to the local geodesic frame, equation (1.17) reduces to

$$\partial_\nu T^{\mu\nu} = -\partial^\mu\phi\left[\frac{\partial\mathscr{L}_m}{\partial\phi} - \partial_\nu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\nu}}\right] \tag{1.18}$$

and upon integration over the volume of the system:

$$\frac{d}{dx^0}P^\mu = -\int \partial^\mu\phi\left[\frac{\partial\mathscr{L}_m}{\partial\phi} - \partial_\nu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\nu}}\right]d^3x \tag{1.19}$$

where $P^\mu = \int T^{\mu 0}\,d^3x$ is the momentum. This indicates that the system can experience an acceleration with respect to the local inertial frame unless we require that the integral on the right vanish identically.

The restriction†

$$\int \partial^\mu\phi\left[\frac{\partial\mathscr{L}_m}{\partial\phi} - \partial_\nu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\nu}}\right]d^3x = 0 \tag{1.20}$$

forbids couplings of the type 'mass varying with ϕ', i.e., a term $m^2c^2f_2(\phi)\Phi^2$ such as appears in equation (1.6). This shows that it is unimportant which particle we use to define a unit of mass. We remark that our assumptions exclude a scalar-tensor theory constructed by Schwinger (1970) in which the function f_2 of equation (1.6) depends on both ϕ and R.

Also forbidden are couplings of the type 'coupling "constant" varying with ϕ', i.e., the electromagnetic, strong and weak coupling constants cannot be functions of ϕ. Direct interaction of photons with ϕ (a term $h(\phi)F_{\mu\nu}F^{\mu\nu}$ in the Lagrangian) is also excluded. It is therefore natural to suppose that $\mathscr{L}_m$ has no dependence at all on ϕ. This leads to theories of the type considered by Wagoner (1970) in which there appear only two

† The term $\partial^\mu\phi$ can actually be omitted from the integrand since, within our approximations, it is a constant.

arbitrary functions of ϕ (corresponding to our $f(\phi)$ and $\lambda(\phi)$). Our arguments give some justification for the 'principle of mutual coupling' postulated by Wagoner. In fact, a simple transformation of variables (introduce a new (metric) tensor field $g'_{\mu\nu}=\psi^2 g_{\mu\nu}$ in equation (1.8) of Wagoner's paper) shows that this principle implies the absence of direct coupling between matter and the scalar field.

2. *Inertial and Gravitational Mass*

The linear approximation to the vacuum field equations that follow from (1.14) and (1.15) is

$$-\tfrac{1}{2}[\gamma_{\mu\nu,\alpha}{}^{,\alpha}-\gamma_{\mu\alpha}{}^{,\alpha}{}_{,\nu}-\gamma_{\nu\alpha}{}^{,\alpha}{}_{,\mu}+\delta_{\mu\nu}\gamma_{\beta\alpha}{}^{,\alpha,\beta}]=$$

$$\frac{f_0'}{f_0}(\xi_{,\mu,\nu}-\delta_{\mu\nu}\xi_{,\alpha}{}^{,\alpha})-\left(\frac{\lambda}{f}\right)_0'\delta_{\mu\nu}\xi-\left(\frac{\lambda}{f}\right)_0(\gamma_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}\gamma_\alpha{}^\alpha+\delta_{\mu\nu}) \quad (2.1)$$

$$\times\,\xi_{,\alpha}{}^{,\alpha}-\frac{\lambda'_0}{\omega}(\phi_0+\xi)-\lambda_0''\phi_0\xi=0 \quad (2.2)$$

where all the contractions are done with the flat space metric $\delta^{\mu\nu}$. The variables $\gamma_{\mu\nu}$ and ξ have been defined by

$$g_{\mu\nu}=\gamma_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}\gamma_\alpha{}^\alpha+\delta_{\mu\nu} \quad (2.3)$$

$$\phi=\phi_0+\xi \quad (2.4)$$

and

$$f_0=f(\phi_0) \quad (2.5)$$

$$f_0'=f'(\phi_0) \quad (2.6)$$

where ϕ_0 is the asymptotic value of the scalar field.

We now require that at large distances from their source both $g_{\mu\nu}$ and ϕ approach their asymptotic values ($\delta_{\mu\nu}$ and ϕ_0, respectively) as $1/r$. This gives a precise definition to what is meant by 'long-range' in hypothesis *i*). Such a behavior of the asymptotic fields is only possible if $\lambda_0=\lambda_0'=\lambda_0''=0$.

Under these conditions the static solutions of equations (2.1) and (2.2) are given by

$$g_{xx}=g_{yy}=g_{zz}=-1-\frac{1}{r}\left(B-2\frac{f_0'}{f_0}A\right)$$

$$g_{xy}=g_{yz}=g_{zx}=0$$

$$g_{00}=1-\frac{B}{r} \quad (2.7)$$

$$\phi=\phi_0+\frac{A}{r} \quad (2.8)$$

where A and B are constants.

Since the total energy momentum tensor $\mathscr{T}_\mu{}^\nu$ can be written as a divergence,† the inertial mass

$$M_I = \int \mathscr{T}_0{}^0 \, d^3x \tag{2.9}$$

can be expressed as a surface integral at infinity. (We assume that our system is static or quasistatic so that time derivatives can be neglected.) The surface integral can be evaluated by using equations (2.7) and (2.8). The result is

$$M_I = \frac{f_0}{2f_0'}\left(B - \frac{f_0'}{f_0}A\right) \tag{2.10}$$

The gravitational mass is simply

$$M_G = B/2G_0 \tag{2.11}$$

which follows from (2.7) if one recalls that test particles may be regarded as moving in a potential $\frac{1}{2}(g_{00}-1)$ which must therefore equal the Newtonian potential $-G_0 M/r$. The value of the gravitational constant has been designated by G_0.

The ratio of inertial to gravitational mass can then be expressed as

$$\frac{M_I}{M_G} = \frac{f_0 G_0}{f_0'(1 + A/2M_I)} \tag{2.12}$$

This ratio can be a universal constant only if $A/2M_I$ is a universal constant. We will see that this is not the case. By combining equations (1.14) and (1.15) the exact field equation satisfied by ϕ can be written

$$\Box\phi = \left[\frac{2\omega}{\phi} + \frac{3(f')^2}{f}\right]^{-1}\left[8\pi\frac{f'}{f}T + \phi_{,\mu}\phi^{,\mu}\left(\frac{\omega}{\phi^2} - \frac{\omega}{\phi}\frac{f'}{f} - \frac{3}{f}f''f'\right)\right.$$
$$\left. - \frac{4\lambda f'}{f} + 2\lambda' + 16\pi\left(\mathscr{L}_m' - \partial_\mu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\mu}}\right)\right] \tag{2.13}$$

If we integrate both sides of this equation over all space, the left side can be converted into a surface integral with the result

$$4\pi A = \int \left[\frac{2\omega}{\phi} + \frac{3(f')^2}{f}\right]^{-1}\left[8\pi\frac{f'}{f}T + \phi_{,\mu}\phi^{,\mu}\left(\frac{\omega}{\phi^2} - \frac{\omega}{\phi}\frac{f'}{f} - \frac{3}{f}f''f'\right)\right.$$
$$\left. - \frac{4\lambda f'}{f} + 2\lambda' + 16\pi\left(\mathscr{L}_m' - \partial_\mu\frac{\partial\mathscr{L}_m}{\partial\phi_{,\mu}}\right)\right]\sqrt{(-g)}\,d^3x \tag{2.14}$$

First, we look at the case in which the gravitational energy can be neglected. The constant A can be approximated as [using (1.20)]

$$A = 2\zeta \int T d^3x \tag{2.15}$$

where

$$\zeta = \frac{f_0'}{f_0}\left[\frac{2\omega}{\phi_0} + \frac{3(f_0')^2}{f_0}\right]^{-1} \tag{2.16}$$

† Details (for the special case of the Brans-Dicke theory) may be found in Ohanian (1971).

and

$$\frac{M_I}{M_G} = \frac{f_0 G_0}{f_0'\left(1 + \zeta \int T d^3 x \Big/ \int T_0{}^0 d^3 x\right)} \tag{2.17}$$

For static systems in which $\partial_\mu T_\nu{}^\mu = 0$ one has $\int T d^3 x = \int T_0{}^0 d^3 x$ and therefore

$$\frac{M_I}{M_G} = \frac{f_0 G_0}{f_0'(1 + \zeta)} \tag{2.18}$$

The ratio M_I/M_G is then a universal constant, as it should be. By definition of G_0, this universal constant is unity, i.e.,

$$G_0 = \frac{f_0'}{f_0}(1 + \zeta) \tag{2.19}$$

Using this expression for G_0 we obtain M_I/M_G for any arbitrary system as

$$\frac{M_I}{M_G} = \left[1 + \frac{A - 2\zeta M_I}{2M_I(1 + \zeta)}\right]^{-1} \tag{2.20}$$

where A is given by equation (2.14). The term $A - 2\zeta M_I$ is of the order of magnitude of the gravitational energy and equation (2.20) shows that M_I/M_G will differ from unity by a term of the order of gravitational energy divided by M_I.

3. *Conclusions*

In regard to the hypothesis (i), (ii) and (iii) we can say that the first is plausible, the second unavoidable and the third is a reasonable generalization of some very precise experimental results. Our conclusions about the restrictions on the couplings between the scalar field and matter and about the violation of the principle of equivalence must hold in almost any conceivable scalar-tensor theory. If the measurement of the relative acceleration of earth and moon by laser ranging (Thorne, 1970) should show that the earth's gravitational and inertial masses are equal, this would not only be evidence against the Brans-Dicke theory, but also against almost any other kind of scalar-tensor theory.

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